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# FLOW AND HEAT TRANSFER IN STEADY LAMINAR COMPRESSIBLE BOUNDARY LAYER SWIRLING FLOW IN A NOZZLE

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**Abstract**—The axisymmetric steady laminar compressible boundary layer swirling flow of a gas with variable properties in a nozzle has been investigated. The partial differential equations governing the non-similar flow have been transformed into new co-ordinates having finite ranges by means of a transformation which maps an infinite range into a finite one. The resulting equations have been solved numerically using an implicit finite-difference scheme. The computations have been carried out for compressible swirling flow through a convergent conical nozzle. The results indicate that the swirl exerts a strong influence on the longitudinal skin friction, but its effect on the tangential skin friction and heat transfer is comparatively small. The effect of the variation of the density–viscosity product across the boundary layer is appreciable only at low-wall temperature. The results are in good agreement with those of the local-similarity method for small values of the longitudinal distance.

## NOMENCLATURE

$a$, scale factor in equation (5);
$b$, constant;
$C_f, \bar{C}_f$, skin-friction coefficients along $\xi$ and $\eta$ directions, respectively;
$f$, dimensionless stream function;
$f_w$, mass-transfer parameter defined by equation (7);
$f'$, dimensionless longitudinal velocity, $u/u_e$;
$F''_w, S'_w$, surface skin-friction parameters along $\xi$ and $\eta$ directions defined by equation (4d);
$g$, dimensionless total enthalpy, $H/H_e$;
$g_w$, cooling parameter, $H_w/H_e$;
$G'_w$, surface heat-transfer parameter defined by equation (4d);
$h$, enthalpy;
$H$, total enthalpy, $h+(u^2+v^2)/2$;
$L$, length of the nozzle measured along $\xi$ direction;
$Pr$, Prandtl number;
$q$, heat-transfer rate;
$r$, surface or body radius;
$s$, dimensionless swirl velocity, $v/v_e$;
$St$, Stanton number;
$T$, temperature;
$u, v, w$, velocity components along $\xi$, $\eta$, $\zeta$ directions, respectively;
$u_e^2/2H_e$, $v^2/2H_e$, dissipation parameters;
$X, Z$, transformed co-ordinates defined by equation (3a);
$y$, transformed normal distance defined by equation (5).

Greek symbols

$\alpha, \beta$, swirl and longitudinal acceleration parameters defined by equation (3d);
$\Gamma$, constant circulation;
$\lambda$, semi-vertical angle of the nozzle;
$\mu$, viscosity;
$\xi, \eta, \zeta$, longitudinal, tangential and normal directions, respectively;
$\bar{\xi}$, dimensionless longitudinal distance;
$\rho$, density;
$\tau_\xi, \tau_\eta$, shear stress along $\xi$ and $\eta$ directions, respectively;
$\phi$, density–viscosity product ratio defined by equation (3c);
$\omega$, exponent in the power-law variation of viscosity.

Superscripts

$'$, prime denotes differentiation with respect to $Z$.

Subscripts

$e$, denotes condition at the edge of the boundary layer;
$i$, denotes inlet condition;
$w$, denotes condition at the surface $\zeta = Z = 0$.

## 1. INTRODUCTION

SWIRLING flows are encountered in rockets (especially spin-stabilized rockets), plasma jets, vortex valves, jet engines, industrial furnaces (vortex burners), twisted-tape swirl generators, swirl atomizers, and many other types of machinery. They have been the subject of numerous theoretical and experimental studies. Lewellen [1] and Murthy [2] have made an extensive survey of swirling flows and their applications. Swirling flows through nozzles are generally compressible, three-dimensional and non-similar in character and their exact prediction requires the solution of a set of coupled non-linear partial differential equations. Such equations are rather difficult to solve and require considerable manpower and computer time. Hence most of the investigators have obtained the solution of the governing equations under certain simplifying assumptions.

The velocity profiles in the laminar boundary layer in swirling flow of an incompressible viscous fluid in a convergent nozzle was first investigated by Taylor [3] who considered the case of a dominant tangential flow superimposed upon a secondary axial flow. Recently, Houlihan and Hornstra [4] have studied the above problem by taking into account the effect of boundary-layer growth upon the tangential and axial velocities in the free-stream. They found that, in contrast to the results obtained by Wilks [5], no "velocity overshoot" exists within the boundary layer. Back [6] has studied the effect of swirl on the flow and heat transfer in an axisymmetric compressible low-speed laminar boundary layer for a gas with constant properties ($\rho \propto T^{-1}$, $\mu \propto T$, $Pr = 1$) flowing in a channel of variable cross-sectional area and also in a convergent conical nozzle. The partial differential equations governing the flow were reduced to ordinary differential equations by similarity transformations and then solved by quasi-linearization technique in conjunction with the concept of local similarity. Muthanna and Nath [7] have extended the work of Back [6] to include variable gas properties, non-unity Prandtl number and mass transfer.

In this paper, the non-similar problem has been studied taking into account realistic gas properties ($\rho \propto T^{-1}$, $\mu \propto T^{\omega}$, $Pr = 0.7$). The partial differential equations governing the non-similar flow have been transformed into new co-ordinates with finite range by means of a transformation which maps an infinite interval into a finite one and the resulting equations have been solved numerically using an implicit finite-difference scheme [8–9]. The numerical computations have been carried out for compressible swirling flow through a convergent conical nozzle.

## 2. GOVERNING EQUATIONS

The governing partial differential equations for laminar compressible boundary layer in swirling flow of a perfect gas with variable properties ($\rho \propto T^{-1}$, $\mu \propto T^{\omega}$, $Pr = 0.7$) over an axisymmetric surface of variable cross-section caused by imposing a free-vortex on longitudinal flow in dimensionless form are [6]

$$(\phi f'')' + ff'' + \beta[(\rho_e/\rho) - f'^2] + \alpha[(\rho_e/\rho) - s^2] = 2X[f'(\partial f'/\partial X) - f''(\partial f/\partial X)] \quad (1a)$$

$$(\phi s')' + fs' = 2X[f'(\partial s/\partial X) - s'(\partial f/\partial X)] \quad (1b)$$

$$(\phi g'/Pr)' + fg' + (u_e^2/2H_e) \times \{2\phi[1-(1/Pr)][f'f'' + (v_e/u_e)^2 ss']\}' = 2X[f'(\partial g/\partial X) - g'(\partial f/\partial X)]. \quad (1c)$$

The boundary conditions are

$$f(X,0) = f_w, \quad f'(X,0) = s(X,0) = 0, \quad f'(X,\infty) = s(X,\infty) = 1 \quad (2a)$$

$$g(X,0) = g_w, \quad g(X,\infty) = 1 \quad (2b)$$

where

$$Z = (r\rho_e u_e)(2X)^{-1/2}\int_0^{\zeta} (\rho/\rho_e)\,d\zeta, \quad X = \int_0^{\xi} \rho_e\mu_e u_e r^2\,d\xi \quad (3a)$$

$$\rho_e/\rho = \{g - (u_e^2/2H_e)[f'^2 + (v_e/u_e)^2 s^2]\}/[1 - (u_e^2 + v_e^2)/2H_e] \quad (3b)$$

$$\phi = \rho\mu/\rho_e\mu_e = (\{g - (u_e^2/2H_e)[f'^2 + (v_e/u_e)^2 s^2]\}/[1 - (u_e^2 + v_e^2)/2H_e])^{\omega-1} \quad (3c)$$

$$\alpha = (2X/r)(-dr/dX)(v_e/u_e)^2, \quad \beta = (2X/u_e)(du_e/dX). \quad (3d)$$

It may be remarked that $\omega = 1$ gives the constant density–viscosity product simplification ($\phi = 1$), $\omega = 0.7$ is appropriate for low-temperature flows while $\omega = 0.5$ may be regarded as a limiting value for high-temperature flows [10]. We have considered the Prandtl number $Pr$ as a constant because in most problems involving air as the working fluid, its variation in the boundary layer is small.

The skin-friction coefficients along the longitudinal and tangential directions are, respectively, given by [6]

$$C_f = 2\tau_\xi/\rho_e u_e^2 = [(2/X)^{1/2} r\mu_e]F''_w \quad (4a)$$

$$\bar{C}_f = 2\tau_\eta/\rho_e u_e^2 = [(2/X)^{1/2}(v_e/u_e) r\mu_e]S'_w. \quad (4b)$$

Similarly, the heat-transfer coefficient in the form of Stanton number can be expressed as [6]

$$St = q/[(H_e - H_w)\rho_e u_e] = r\mu_e(2X)^{-1/2}G'_w \quad (4c)$$

where

$$F''_w = \phi_w f''_w, \quad S'_w = \phi_w s'_w, \quad G'_w = \phi_w g'_w/[Pr(1-g_w)]. \quad (4d)$$

## 3. TRANSFORMATION TO FINITE CO-ORDINATES

The governing equations (1) are transformed to a new system of co-ordinates wherein the infinite range of integration $(0, \infty)$ on $Z$ is replaced by a finite range $(0, 1)$. The transformation is given by [8]

$$y = 1 - \exp(-aZ) \quad (5)$$

where $a$ is a scaling factor to provide an optimum distribution of nodal points across the boundary layer [8]. Defining $(\partial f/\partial Z) = F$, $a(1-y) = z$ and changing the variable $Z$ to $y$ by means of (5), equations (1) become

$$\phi z^2 F_{yy} + z(z\phi_y - a\phi + f)F_y + \beta[(\rho_e/\rho) - F^2] + \alpha[(\rho_e/\rho) - s^2] = 2X(FF_X - zf_X F_y) \quad (6a)$$

$$\phi z^2 s_{yy} + z(z\phi_y - a\phi + f)s_y = 2X(Fs_X - zf_X s_y) \quad (6b)$$

$$\begin{aligned}(\phi/Pr)z^2 g_{yy} &+ z[(z\phi_y - a\phi)/Pr + f]g_y \\ &- [1-(1/Pr)](u_e^2/H_e)\{(\alpha+\beta)(\rho_e/\rho)F \\ &- F(\beta F^2 + \alpha s^2) + zF_y(fF - z\phi F_y) \\ &- (v_e/u_e)^2 z(z\phi s_y - fs)s_y] \\ &= 2X\{(Fg_X - zF_x g_y) - [F(FF_X - zf_X F_y) \\ &+ (v_e/u_e)^2 s(Fs_X - zf_X s_y)] \\ &\times [(1-1/Pr)(u_e^2/H_e)]\}\end{aligned} \quad (6c)$$

where

$$f = \int_0^y \frac{F}{a(1-y)}\,dy + f_w, \quad f_w = -(2X)^{-1/2}\int_0^{\xi}(\rho w)_w r\,d\xi. \quad (7)$$

The boundary conditions (2) are now transformed to

$$F(X,0) = s(X,0) = 0, \quad g(X,0) = g_w, \quad F(X,1) = s(X,1) = g(X,1) = 1 \quad (8)$$

The parameters $f_w''$, $s_w'$ and $g_w'$ occurring in (4) can be expressed as

$$f_w'' = a(F_y)_w, \quad s_w' = a(s_y)_w, \quad g_w' = a(g_y)_w. \quad (9)$$

It may be noted that $\rho_e/\rho$ and $\phi$ which occur in (6) are given by (3b) and (3c), respectively.

## 4. METHOD OF SOLUTION

The axisymmetric boundary-layer flow with swirl for any arbitrary body may be studied provided the variation of $\alpha$, $\beta$, $u_e$ and $v_e$ with the longitudinal distance $X$ (or $\xi$) is known. The governing equations (6) under conditions (8) may be converted into a set of implicit finite-difference equations and the resulting linear tridiogonal matrix equations may be solved by the use of the Thomas algorithm [11]. This method is essentially the same as that employed by Marvin and Sheaffer [8] and Vimala and Nath [9], except that the fully implicit finite-difference scheme instead of Crank–Nicholson finite-difference scheme and Thomas algorithm instead of the algorithm given by Varga [12] have been used. Since detailed description of the method along with its application to boundary-layer equations without swirl has been given in [8, 9], for the sake of brevity, it is not described here. It may be noted that the use of Thomas algorithm for the solution of tridiogonal matrix equations takes less computer time as compared to the matrix inversion method [11].

## 5. RESULTS AND DISCUSSION

The solutions of equations (6) under conditions (8) for both swirling ($\alpha > 0$) and non-swirling [$\alpha = 0$, $(v_e/u_e)_i = 0$] flows have been obtained on an IBM 360/44 computer in the case of a conical convergent nozzle. For this case, we have [6]

$$u_e = b/r^2, \quad v_e = \Gamma/r, \quad r = L(1-\bar{\xi})\sin\lambda, \quad \bar{\xi} = \xi/L \quad (10a)$$

$$\alpha = 2(1-\bar{\xi})\bar{\xi}(v_e/u_e)_i^2, \quad \beta = 4\bar{\xi}/(1-\bar{\xi}) \quad (10b)$$

$$u_e^2/H_e = (u_e^2/H_e)_i(1-\bar{\xi})^{-4}, \quad v_e^2/H_e = (v_e^2/H_e)_i(1-\bar{\xi})^{-2} \quad (10c)$$

$$(v_e/u_e)^2 = (v_e/u_e)_i^2(1-\bar{\xi})^2, \quad X(\partial/\partial X) = \bar{\xi}(\partial/\partial\bar{\xi}). \quad (10d)$$

The computations have been carried out for $(v_e/u_e)_i^2 = 1, 10$; $(v_e^2/H_e)_i = 0.01, 0.1$ and $(u_e^2/H_e)_i = 0.01$. Here $Pr$ is taken as 0.7 and $a = 0.5$. The step size $\Delta y = 0.005$ and $\Delta\bar{\xi} = 0.05$ have been used for computation and further reduction in them does not change the results up to the third decimal place. The effect of mass transfer is not considered in the present case (i.e. $f_w = 0$). It is evident from (10c) that, for given $(u_e^2/H_e)_i$ and $(v_e^2/H_e)_i$, $u_e^2/H_e$ and $v_e^2/H_e$ rapidly increase with $\bar{\xi}$ and they tend to infinity as $\bar{\xi} \to 1$. It can also be seen that $1-(u_e^2+v_e^2)/2H_e$ [which occurs in the denominator of (6)] tends to zero for some critical value $(\bar{\xi})^* < 1$, depending on the magnitude of $(u_e^2/H_e)_i$ and $(v_e^2/H_e)_i$. Hence, the solutions are not valid beyond the critical values of $\bar{\xi}$. The range of the validity of the solutions decreases as $(u_e^2/H_e)_i$ and $(v_e^2/H_e)_i$ increase. It is clear that the dissipation terms $u_e^2/H_e$ and $v_e^2/H_e$ affect the solutions significantly. It may be remarked that Back [6] has investigated the above problem for $Pr = 1$ by neglecting the variation of the density–viscosity product across the boundary layer and the dissipation terms. Using the concept of local similarity, he solved the governing ordinary differential equations by the technique of quasi-linearization. In order to compare our results with those obtained by using local-similarity method [6], we have also solved the governing equations using local-similarity concept for $Pr = 0.7$ and $\omega = 1, 0.7, 0.5$ taking into account the effect of viscous dissipation and employing the implicit finite-difference scheme as discussed earlier.

Although the numerical computations were carried out for several values of the parameters, only some representative velocity (both longitudinal and swirl velocities) and total enthalpy profiles are depicted in Figs. 1–3. It is evident from these figures that the profiles $f'$, $s$ and $g$ become more steep when $\omega$ or $\bar{\xi}$ increases. It may be noted that there is no "velocity overshoot" in longitudinal velocity profiles. Similar behaviour has been observed by Houlihan and Hornstra [4] for the incompressible case. As can be seen from the figures, the effects of $\omega$ and $\bar{\xi}$ on the velocity and total enthalpy profiles are quite significant.

The variation of $F_w''$, $S_w'$ and $G_w'$ with $\bar{\xi}$ is displayed in Figs. 4–9. Figures 4–9 also contain some representative results obtained by the local-similarity method. In general, for a given value of $\omega$, $g_w(0 < g_w < 1)$ and $(u_e^2/H_e)_i$, $F_w''$, $S_w'$ and $G_w'$ increase at every point of the convergent section of the nozzle except at the inlet (i.e. when $\bar{\xi} = 0$) when $(v_e/u_e)_i$ or $(v_e^2/H_e)_i$ increases. Similar behaviour is also observed when $\omega$ decreases

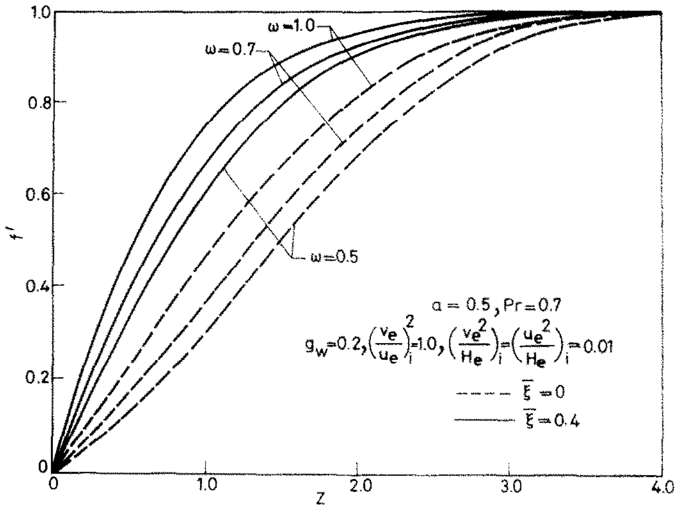


FIG. 1. Longitudinal velocity profiles.

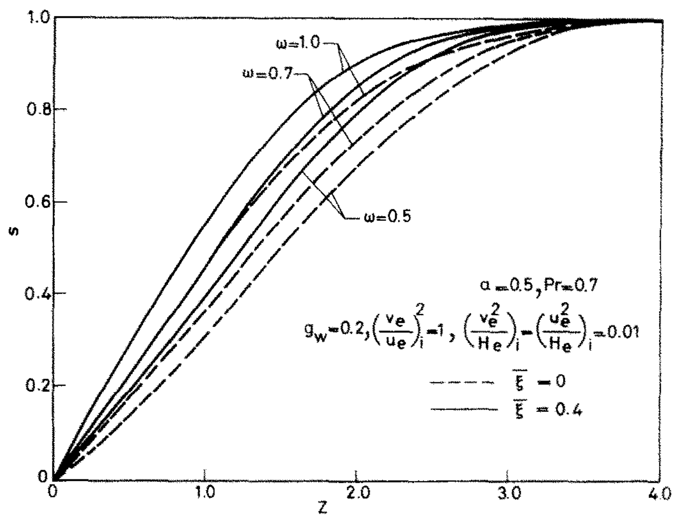


FIG. 2. Swirl velocity profiles.

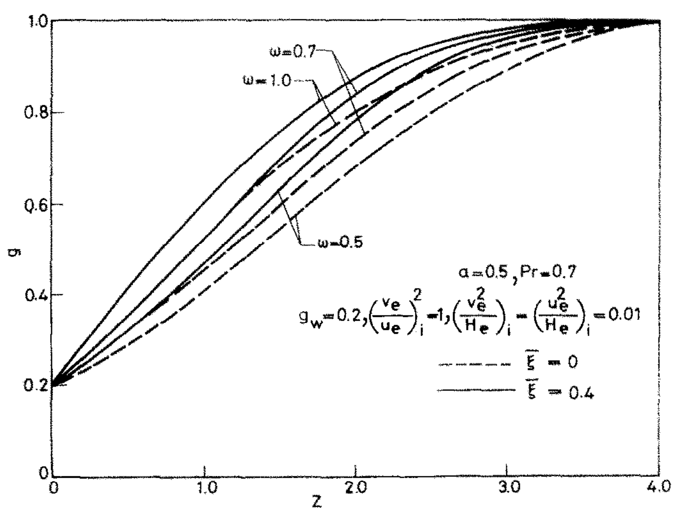


FIG. 3. Total enthalpy profiles.

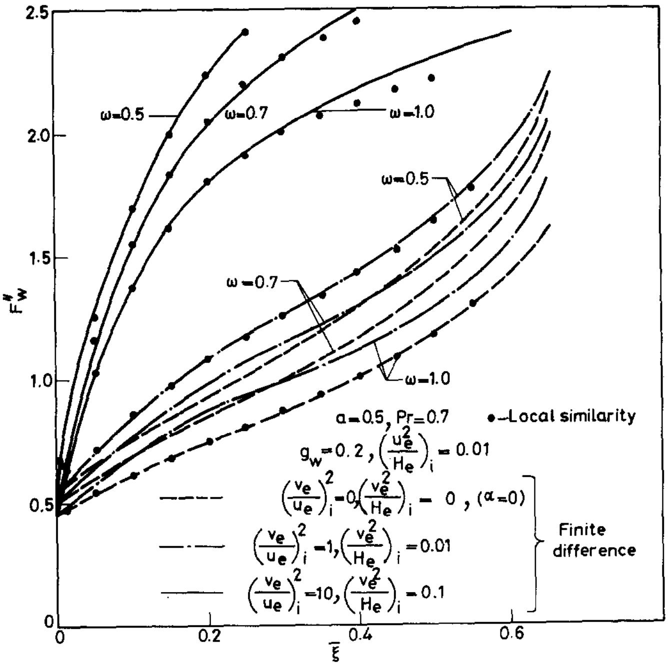

FIG. 4. Variation of longitudinal skin friction with $\bar{\xi}$ ($g_w = 0.2$).

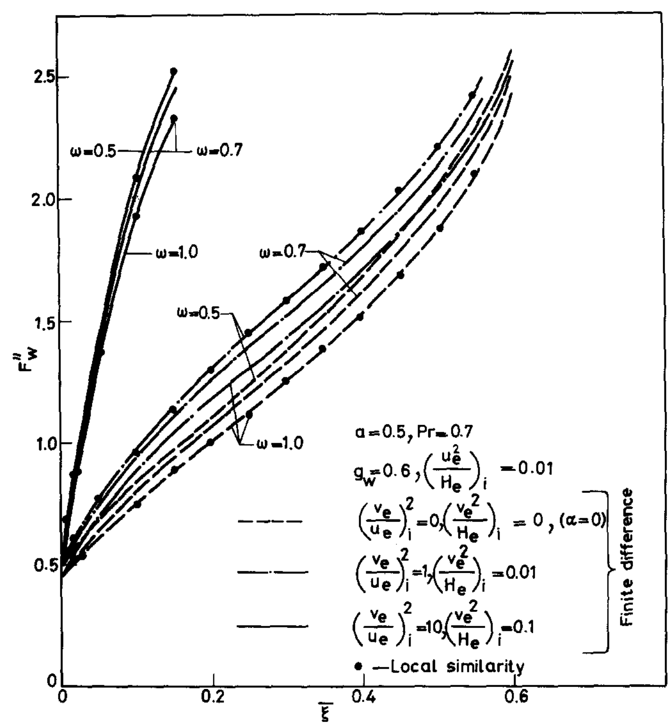

FIG. 5. Variation of longitudinal skin friction with $\bar{\xi}$ ($g_w = 0.6$).

or $g_w$ increases. It may be mentioned that, as $\omega$ decreases, the parameters $f''_w$, $s'_w$ and $g'_w$ which occur in skin-friction and heat-transfer coefficients [see equations (4)] decrease, but $\phi_w$ [see equation (3c)] increases. Consequently, $F''_w$, $S'_w$ and $G'_w$ [see equation (4d)] increase as $\omega$ decreases. It is also seen that for given value of $\alpha$, $F''_w$ increases as $\bar{\xi}$ increases but $G'_w$ increases with $\bar{\xi}$ till a certain value and then it begins to decrease. On the other hand, the behaviour of $S'_w$ is similar to that of $G'_w$ when $(v_e/u_e)_i > 1$, but it behaves like $F''_w$ when $(v_e/u_e)_i = 1$. The results indicate that $\alpha$ exerts a strong influence on $F''_w$, but its effect on $S'_w$ and $G'_w$ is comparatively small. It is further observed that the effect of $\alpha$ on $F''_w$, $S'_w$ and $G'_w$ near the inlet region is

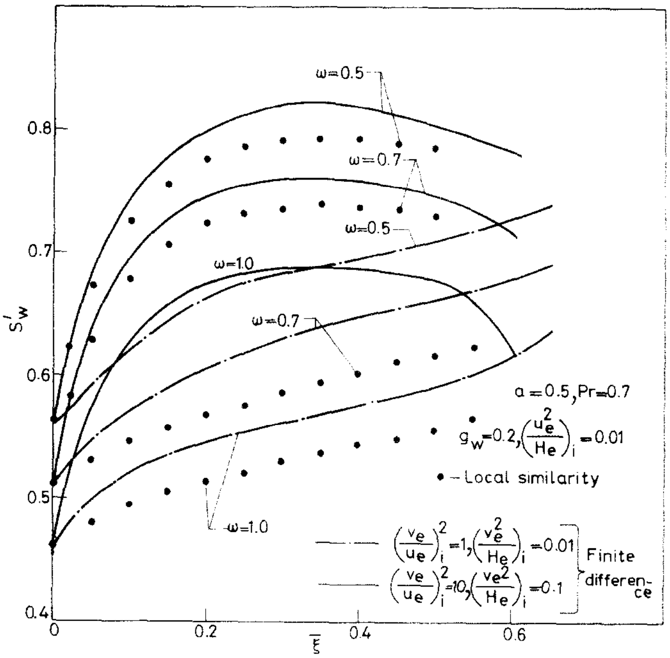


FIG. 6. Variation of tangential skin friction with $\bar{\xi}$ ($g_w = 0.2$).

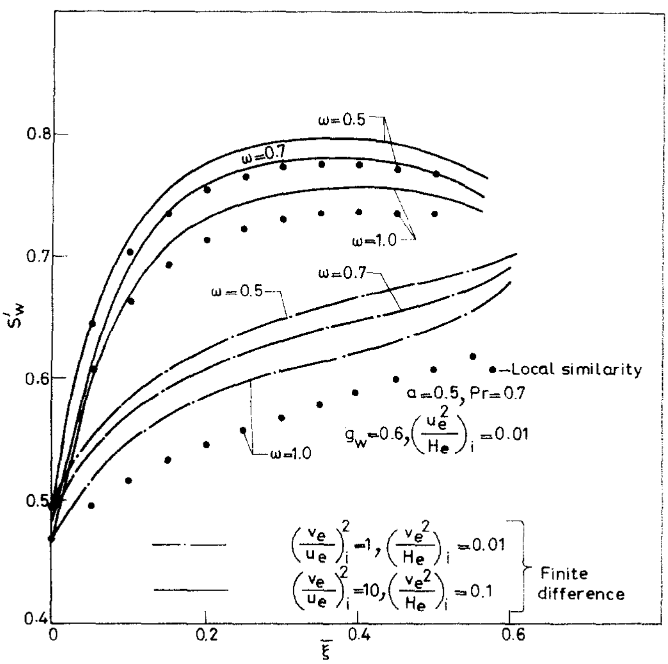


FIG. 7. Variation of tangential skin friction with $\bar{\xi}$ ($g_w = 0.6$).

less as compared to other section of the nozzle. It may be remarked that the effect of variation of $\omega$ on $F''_w$, $S'_w$ and $G'_w$ is more pronounced when $g_w = 0.2$ than when $g_w = 0.6$. Therefore, it can be concluded that the linear viscosity–temperature relation is not a good approximation at low-wall temperatures for calculating skin friction and heat transfer. It may be remarked that our results for $\bar{\xi} = 0$ (similar solution) are the same as those obtained by Back [6] and Muthanna and Nath [7] using quasi-linearization and parametric differentiation techniques, respectively. It is evident from Figs. 4–9 that the skin-friction and heat-transfer results obtained by local-similarity method are in good agreement with the finite-difference results when $\bar{\xi}$ is small, but for skin friction in the longitudinal direction, good agreement between the two methods is observed

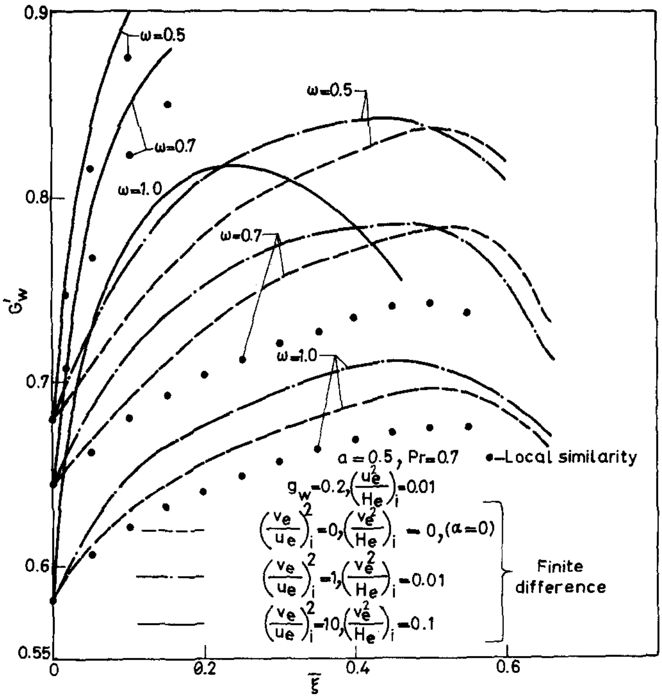


FIG. 8. Variation of heat transfer with $\xi$ ($g_w = 0.2$).

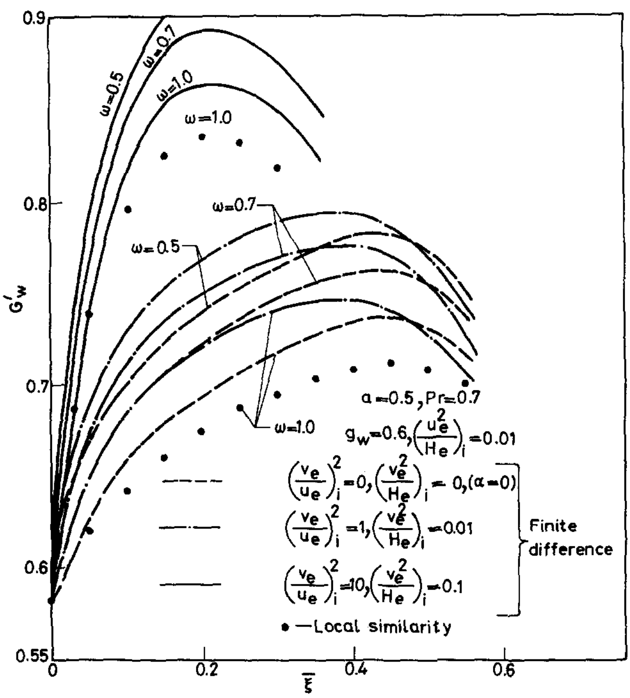


FIG. 9. Variation of heat transfer with $\xi$ ($g_w = 0.6$).

even for large $\xi$. It can be seen from Figs. 4–9 that the local-similarity method underestimates the heat transfer and skin friction in the tangential direction (obtained by finite-difference scheme) along the convergent section of the nozzle except in the inlet region.

## 6. CONCLUSIONS

The longitudinal skin friction is strongly dependent on swirl whereas the dependence of heat transfer and tangential skin friction on swirl is comparatively weak. The effect of the variation of the density–viscosity product across the boundary layer is pronounced only at low-wall temperature indicating that the linear viscosity–temperature relation is not a good approximation at low-wall temperature for calculating skin friction and heat transfer. The results are found to be in good agreement with those obtained by local-similarity method for small values of longitudinal distance.

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## ECOULEMENT ET TRANSFERT THERMIQUE DANS UN MOUVEMENT TOURBILLONNAIRE PERMANENT DANS UNE TUYERE AVEC COUCHE LIMITE LAMINAIRE ET COMPRESSIBLE

**Résumé**—On étudie un écoulement axisymétrique et tourbillonnaire de gaz à propriétés variables dans une tuyère avec couche limite laminaire et compressible. Les équations aux dérivées partielles sont transformées dans de nouvelles coordonnées ayant un domaine fini, à l'aide d'une transformation qui ramène un domaine infini en un autre fini. Les équations résultantes ont été résolues numériquement en utilisant une procédure implicite aux différences finies. Les calculs ont été faits pour un écoulement compressible et tourbillonnaire à travers une tuyère conique convergente. Les résultats indiquent que le tourbillon exerce une forte influence sur le frottement longitudinal à la paroi, mais son effet sur le frottement tangentiel et sur le transfert thermique est comparativement faible. L'effet de la variation du produit masse volumique–viscosité dans la couche limite est appréciable seulement aux faibles températures de paroi. Les résultats sont en bon accord avec ceux de la méthode de la similitude locale pour les faibles valeur de la distance longitudinale.

## STATIONÄRE STRÖMUNG UND WÄRMEÜBERGANG IN DER LAMINAREN, KOMPRESSIBLEN GRENZSCHICHT IN EINER DÜSE MIT DRALLSTRÖMUNG

**Zusammenfassung**—Die achsensymmetrische stationäre Drallströmung eines Gases mit laminarer kompressibler Grenzschicht und veränderlichen Stoffwerten in einer Düse wurde untersucht. Die partiellen Differentialgleichungen, welche die nicht ähnliche Strömung bestimmen, wurden in neue Koordinaten mit endlichen Bereichen transformiert, wobei eine Transformation verwendet wurde, die einen unendlichen auf einen endlichen Bereich abbildet. Die sich daraus ergebenden Gleichungen wurden numerisch mit Hilfe eines impliziten finiten Differenzenverfahrens gelöst. Die Berechnungen wurden ausgeführt für kompressible Drallströmung durch eine konvergente konische Düse. Die Ergebnisse zeigen, daß der Drall einen großen Einfluß auf die Wandreibung in Längsrichtung hat, jedoch sein Einfluß auf die tangentiale Wandreibung und den Wärmeübergang verhältnismäßig klein ist. Die Auswirkung der Variation des Produktes aus Dichte und Zähigkeit quer zur Grenzschicht ist nur bei niedrigen Wandtemperaturen von Bedeutung. Die Ergebnisse zeigen für kleine Werte des Längsabstandes gute Übereinstimmung mit denen der örtlichen Ähnlichkeitsmethode.